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Spin-up from a rotating disk

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An analytical–numerical analysis of the fluid motion induced by an impulsive change in the angular velocity of a rotating disk is presented. Separate solutions are derived for the early development and for the approach to the ultimate state. The superposed flow is found to have variations on two length scales, and is characterized by the generation and propagation of shear-type wave trains. These decaying wave trains ultimately lead to the new von Kármán state corresponding to the changed angular velocity of the disk.

1. Introduction

In this paper we consider fluid motion induced by an impulsive change in the angular velocity of a disk rotating in contact with an incompressible viscous fluid. The physical situation will be as follows: the fluid occupying the region $z \geqslant 0$ is in steady motion owing to the uniform rotation (at angular velocity Ω) of a disk at $z = 0$. At time $t = 0$, the angular velocity of the disk is changed by a small amount $\epsilon\Omega$. Our aim now is to describe the transition of the fluid to the new steady state corresponding to the new uniform rotation of the disk with angular velocity $\Omega(1+\epsilon)$. Here we confine our attention to small perturbations in the basic von Kármán steady state and construct two separate solutions which are valid during the initial and final stages of the spin-up process respectively. A general treatment of the oscillatory character of the superposed fluid motion is also presented.

The situation considered here is evidently analogous to the time-dependent flow of a rotating fluid discussed by Greenspan & Howard (1963), although in their case there are two infinite disks initially in rigid-body rotation with the fluid. They have proved that the ultimate state is reached through small amplitude decaying inertial oscillations of twice the frequency of basic rotation. Chawla (1972) has shown that the spin-up of a rotating fluid admits the generation and propagation of decaying shear-type wave trains. The spin-up problem for a rotating disk in an infinite fluid is essentially difficult because the basic steady state is not known analytically. Hence, unlike Greenspan & Howard (1963), we may not be able to obtain simple solutions. Nevertheless, the method of analysis employed in this paper brings out a representative physical mechanism of the system.

A secondary motivation of the present paper is to examine the validity of Benton's (1966) conjecture for impulsively started rotating flows in general.

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Benton (1966, see also Thiriot 1940; Nigam 1951) has studied the evolution of the von Kármán flow when the fluid motion is started impulsively from rest. He has surmised (without proof of course) that, in analogy with the Greenspan–Howard flow, the von Kármán steady state is reached through small amplitude decaying oscillations of frequency 2Ω. The present formulation, however, reveals that the frequency of the superposed oscillatory motion is too small to be effective before the ultimate state is achieved. We find that diffusion is the dominant feature throughout the spin-up process. The interaction of diffusion with the basic von Kármán axial inflow generates shear-type aperiodic wave trains which ultimately decay, leading to the new von Kármán state.

2. Mathematical formulation

We take cylindrical co-ordinates (r, θ, Z) in a non-rotating frame of reference, with T as the time. Let $\mathbf{V}$ and p respectively be the velocity vector and the pressure when the angular velocity of the disk is changed (at $T = 0$) from Ω to $\Omega(1+\epsilon)$. Consistent with the continuity equation, we define

$$V = -\tfrac{1}{2}r\Omega H_z\hat{\mathbf{r}} + r\Omega G\hat{\boldsymbol{\theta}} + (\nu\Omega)^{\frac{1}{2}}H\hat{\mathbf{z}}, \tag{2.1}$$

with
$$p = -\rho\nu\Omega P, \quad z = (\Omega/\nu)^{\frac{1}{2}}Z, \quad \tau = \Omega T, \tag{2.2}$$

where H, G and P are functions of z and τ only and $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{z}}$ are unit vectors along the r, θ and z directions respectively. In addition, a suffix denotes a partial derivative, and ρ and ν are respectively the fluid density and kinematic viscosity. Substitution of (2.1) and (2.2) into the axisymmetric governing equations leads to

$$H_{zzz} - H_{z\tau} - HH_{zz} + \tfrac{1}{2}H_z^2 - 2G^2 = 0, \tag{2.3}$$

$$G_{zz} - G_\tau - HG_z + GH_z = 0. \tag{2.4}$$

Appropriate initial and boundary conditions for the time-dependent fluid flow are

$$H = H^0, \quad G = G^0 \quad \text{at} \quad \tau = 0, \tag{2.5}$$

$$\left.\begin{aligned} H = 0, \quad H_z = 0, \quad G = 1+\epsilon \quad \text{on} \quad z = 0 \\ H_z \to 0, \quad G \to 0 \quad \text{as} \quad z \to \infty \end{aligned}\right\} \quad \text{for} \quad \tau \geqslant 0, \tag{2.6}$$

where H^0 and G^0 are the von Kármán functional forms associated with the basic steady velocity field. We seek the solutions of the system (2.3)–(2.6) when $\epsilon \ll 1$, and write

$$H(z,\tau) = H^0(z) + \epsilon H^1(z,t), \quad G(z,\tau) = G^0(z) + \epsilon G^1(z,t), \tag{2.7}$$

with
$$t = c^2\tau, \tag{2.8}$$

where $c = 0{\cdot}884$ is the steady von Kármán inflow far away from the disk. Substituting (2.7) and (2.8) in (2.3) and (2.4) and neglecting terms of order ϵ^2, we get

$$H^0_{zzz} - H^0H^0_{zz} + \tfrac{1}{2}H^{02}_z - 2G^{02} = 0, \tag{2.9a}$$

$$G^0_{zz} - H^0G^0_z + G^0H^0_z = 0, \tag{2.9b}$$

with
$$H^0(0) = 0, \quad H^0_z(0) = 0, \quad G^0(0) = 1, \quad H^0_z(\infty) = 0 = G^0(\infty), \tag{2.9c}$$

$$H^1_{zzz} - c^2 H^1_{zt} - H^1 H^0_{zz} - H^0 H^1_{zz} + H^0_z H^1_z - 4G^0 G^1 = 0, \tag{2.10a}$$

$$G^1_{zz} - c^2 G^1_t - H^0 G^1_z - H^1 G^0_z + H^0_z G^1 + H^1_z G^0 = 0, \tag{2.10b}$$

$$H^1(z,0) = 0 = G^1(z,0),$$

$$H^1(0,t) = 0, \quad H^1_z(0,t) = 0, \quad G^1(0,t) = 1, \quad H^1_z(\infty,t) = 0, \quad G^1(\infty,t) = 0. \tag{2.10c}$$

The steady-state problem characterized by the set (2.9) has been solved by von Kármán (1921), Cochran (1934), Fettis (1955) and, more recently, by Benton (1966). In the subsequent analysis of the superposed unsteady flow, governed by the differential set (2.10), we shall make use of the steady solution obtained by Benton.

We now take the Laplace transform of the coupled linear system (2.10):

$$H^1_{zzz} - c^2 s H^1_z - H^1 H^0_{zz} - H^0 H^1_{zz} + H^0_z H^1_z - 4G^0 G^1 = 0, \tag{2.11a}$$

$$G^1_{zz} - c^2 s G^1 - H^0 G^1_z - H^1 G^0_z + G^1 H^0_z + G^0 H^1_z = 0, \tag{2.11b}$$

$$\begin{gathered} H^1(0,s) = 0, \quad H^1_z(0,s) = 0, \quad G^1(0,s) = 1/s, \\ H^1_z(\infty,s) = 0, \quad G^1(\infty,s) = 0, \end{gathered} \tag{2.11c}$$

where s is the transform parameter with respect to t and the same symbols are used for the transformed and the untransformed variables.

In order to account for the induced suction in the far field and the particular solution associated with it, we set

$$sH^1(z,s) = -\alpha(s) + \alpha(s) H^0_z/c^2 s + \bar{H}(z,s), \tag{2.12a}$$

$$sG^1(z,s) = \alpha(s) G^0_z/c^2 s + \bar{G}(z,s), \tag{2.12b}$$

where $\alpha(s)/s$ gives the superposed axial flow towards the disk as $z \to \infty$, and $\bar{H}(z,s)$ and $\bar{G}(z,s)$ are given by

$$\bar{H}_{zzz} - c^2 s \bar{H}_z - \bar{H} H^0_{zz} - H^0 \bar{H}_{zz} + H^0_z \bar{H}_z - 4G^0 \bar{G} = 0, \tag{2.13a}$$

$$\bar{G}_{zz} - c^2 s \bar{G} - H^0 \bar{G}_z - \bar{H} G^0_z + \bar{G} H^0_z + G^0 \bar{H}_z = 0, \tag{2.13b}$$

with
$$\left.\begin{aligned} &\bar{H}(0,s) = \alpha(s), \quad \bar{H}_z(0,s) = -\alpha(s) H^0_{zz}(0)/c^2 s, \\ &\bar{G}(0,s) = 1 - \alpha(s) G^0_z(0)/c^2 s, \\ &\bar{H}(\infty,s) = 0 = \bar{G}(\infty,s). \end{aligned}\right\} \tag{2.13c}$$

Following Benton (1966), we now make the change of variable

$$\lambda = e^{-cz} \tag{2.14}$$

and define
$$H^0(z) = -c + ch(\lambda), \quad G^0(z) = c^2 g(\lambda), \tag{2.15a}$$

$$c\bar{H}(z,s) = \lambda^{m-1} f(\lambda,s), \quad \bar{G}(z,s) = \lambda^{m-1} k(\lambda,s), \tag{2.15b}$$

where
$$m = \tfrac{1}{2} + (s + \tfrac{1}{4})^{\frac{1}{2}}. \tag{2.16}$$

Substitution of (2.14)–(2.16) transforms the differential sets (2.9) and (2.13) into

$$\lambda^3 h''' + \lambda^2(2h'' + hh'' - \tfrac{1}{2}h'^2) + \lambda hh' + 2g^2 = 0, \tag{2.17a}$$

$$\lambda^2 g'' + \lambda(hg' - h'g) = 0, \tag{2.17b}$$

with $$h(1) = 1, \quad h'(1) = 0, \quad c^2 g(1) = 1, \quad g(0) = 0 = h(0), \tag{2.17c}$$

and $$\lambda^3 f''' + \lambda^2[2f'' + fh'' + hf'' - f'h'] + \lambda[hf' + h'f] + 4gk$$
$$+ (m-1)[3\lambda^2 f'' + 2\lambda hf' - \lambda h'f] + (m-1)^2[2\lambda f' - 2f + fh] = 0, \tag{2.18a}$$

$$\lambda^2 k'' + \lambda[hk' - kh' + fg' - gf'] + (m-1)[2(\lambda k' - k) + kh - gf] = 0, \tag{2.18b}$$

with $$\left.\begin{aligned} f(1,s) &= c\alpha(s), \\ f'(1,s) + (m-1)f(1,s) &= c\alpha(s)\,h''(1)/s, \\ k(1,s) &= 1 + c\alpha(s)\,g'(1)/s. \end{aligned}\right\} \tag{2.18c}$$

In these equations a prime denotes differentiation with respect to λ. The substitutions (2.15b) for the functions $\bar{H}$ and $\bar{G}$ are chosen so as to satisfy the boundary conditions at $\lambda = 0$ (corresponding to $z = \infty$). Fettis (1955) and Benton (1966) have solved the system (2.17) by means of power-series expansions in λ of the form

$$h(\lambda) = \sum_{n=1}^{\infty} b_n \lambda^n, \quad g(\lambda) = \sum_{n=1}^{\infty} a_n \lambda^n,$$

where a_n and b_n are constants. We note from (2.18) that $f(0) = 0 = k(0)$, so that we can still expand the functions f and k in the form

$$f(\lambda, s) = \sum_{n=1}^{\infty} f_n \lambda^n, \quad k(\lambda, s) = \sum_{n=1}^{\infty} k_n \lambda^n,$$

where f_n and k_n are functions of s. As pointed out by Benton, such series converge more rapidly than the one in z, since the region of interest now is $0 \leqslant \lambda \leqslant 1$.

Immediately after the impulsive change in the angular velocity of the disk, fluid motion develops near the surface of the rotating boundary. The thickness of the superposed flow regime increases with time. The flow functions $\bar{H}$ and $\bar{G}$ given by (2.15b) serve to account for the increasing thickness of the secondary boundary layer as the transformation parameter varies from $s = \infty$ (corresponding to $t = 0$) to $s = 0$ (in the ultimate state).

3. The initial development

The early time behaviour of the flow functions is determined by the corresponding behaviour of the transformed functions for large $|s|$. But we note that $s = -\frac{1}{4}$ is a branch point of the transformed functions. We therefore expand the functions $\alpha(s), f(\lambda, s)$ and $k(\lambda, s)$ in the form

$$\alpha = \sum_{n=0}^{\infty} (s+\tfrac{1}{4})^{-\frac{1}{2}n} \alpha_n, \quad f = \sum_{n=0}^{\infty} (s+\tfrac{1}{4})^{-\frac{1}{2}n} f_n(\lambda), \quad k = \sum_{n=0}^{\infty} (s+\tfrac{1}{4})^{-\frac{1}{2}n} k_n(\lambda), \tag{3.1}$$

where $|s + \frac{1}{4}|$ is large. A few terms of each of the series (3.1) are given by

$$\alpha_0 = 0, \quad f_0(\lambda) = 0, \quad 2(\lambda k_0' - k_0) + hk_0 = 0, \quad k_0(1) = 1, \tag{3.2a–d}$$

$$\alpha_1 = 0, \quad f_1(\lambda) = 0, \quad 2(\lambda k_1' - k_1) + hk_1 = \lambda(k_0 h' - hk_0') - \lambda^2 k_0'', \quad k_1(1) = 0, \tag{3.3a–d}$$

$$\alpha_2 = 0, \quad 2(\lambda f_2' - f_2) + hf_2 = -4gk_0, \quad f_2(1) = 0, \tag{3.4a–c}$$

$$2(\lambda k_2' - k_2) + hk_2 = gf_2 - \lambda(hk_1' - h'k_1) - \lambda^2 k_1'' + \lambda k_1' - k_1 + \tfrac{1}{2}hk_1, \tag{3.4d}$$

$$k_2(1) = 0, \tag{3.4e}$$

$$c\alpha_3 = -f_2'(1), \tag{3.5a}$$

$$2(\lambda f_3' - f_3) + hf_3 = -4gk_1 - 3\lambda^2 f_2'' - 2\lambda hf_2' + \lambda h' f_2 + 2(\lambda f_2' - f_2) + hf_2, \tag{3.5b}$$

$$f_3(1) = -f_2'(1), \tag{3.5c}$$

$$\begin{aligned} 2(\lambda k_3' - k_3) + hk_3 = gf_3 - \lambda^2 k_2'' - \lambda(hk_2' - k_2 h' + f_2 g' - gf_2') \\ + (\lambda k_2' - k_2) + \tfrac{1}{2}(hk_2 - gf_2), \end{aligned} \tag{3.5d}$$

$$k_3(1) = 0. \tag{3.5e}$$

The above sequence of linear ordinary differential equations is evidently uncoupled and the equations can be solved one after the other in the order written. Although there is no particular difficulty in carrying out the solution to any order, we shall confine ourselves to solving the sets (3.2)–(3.4). Evidently their solution depends upon knowledge of the basic von Kármán state.

Employing the Fettis–Benton method of solution, we take the following power-series expressions for g, h, k_0, k_1, f_2 and k_2:

$$\left.\begin{aligned} g = \sum_{i=1}^{\infty} a_i \lambda^i, \quad h = \sum_{i=1}^{\infty} b_i \lambda^i, \quad k_0 = \sum_{i=1}^{\infty} c_i^0 \lambda^i, \\ k_1 = \sum_{i=1}^{\infty} c_i^1 \lambda^i, \quad f_2 = \sum_{i=1}^{\infty} d_i^2 \lambda^i, \quad k_2 = \sum_{i=1}^{\infty} c_i^2 \lambda^i, \end{aligned}\right\} \tag{3.6}$$

where the coefficients c_i^0, c_i^1, c_i^2 and d_i^2 are to be determined and a_i and b_i correspond to the basic steady-state solution and are tabulated by Benton (1966). Substituting (3.6) into (3.2)–(3.5) and equating the coefficients of different powers of λ to zero, we get the following recursion relations:

$$2(i-1)\, c_i^0 = -\sum_{j=1}^{i-1} c_j^0 b_{i-j}, \tag{3.7}$$

$$2(i-1)\, c_i^1 = -i(i-1)\, c_i^0 + \sum_{j=1}^{i-1} [(i-2j)\, c_j^0 b_{i-j} - c_j^1 b_{i-j}], \tag{3.8}$$

$$2(i-1)\, d_i^2 = -\sum_{j=1}^{i-1} [d_j^2 b_{i-j} + 4c_j^0 a_{i-j}], \tag{3.9}$$

$$2(i-1)\, c_i^2 = -(i-1)^2 c_i^1 + \sum_{j=1}^{i-1} [d_j^2 a_{i-j} - c_j^2 b_{i-j} + (i-2j+\tfrac{1}{2})\, c_j^1 b_{i-j}]. \tag{3.10}$$

It is evident from these relations that all the coefficients in the power series (3.6) can be found in terms of c_1^0, c_1^1, d_1^2 and c_1^2 (since a_i and b_i are already known from the basic steady conditions). These are chosen so as to satisfy the boundary conditions at $\lambda = 1$; which give

$$c_1^0 = 2{\cdot}237, \quad c_1^1 = -1{\cdot}057, \quad d_1^2 = 6{\cdot}433, \quad c_1^2 = 5{\cdot}436. \tag{3.11}$$

The term-by-term inversion of (2.12), after substituting from (2.15) and (3.1), yields the following small-time solution:

$$H^1(z,t) = \frac{2f_2(\lambda)}{c}\left[\operatorname{erfc}\frac{cz-t}{2t^{\frac{1}{2}}} + e^{cz}\operatorname{erfc}\frac{cz+t}{2t^{\frac{1}{2}}} - 2\exp\left(\frac{cz}{2} - \frac{t}{4}\right)\operatorname{erfc}\frac{cz}{2t^{\frac{1}{2}}}\right] + O(t^{\frac{3}{2}}), \tag{3.12}$$

$$G^1(z,t) = \tfrac{1}{2}k_0(\lambda)\left[\operatorname{erfc}\frac{cz-t}{2t^{\frac{1}{2}}}+e^{cz}\operatorname{erfc}\frac{cz+t}{2t^{\frac{1}{2}}}\right]+k_1(\lambda)\left[\operatorname{erfc}\frac{cz-t}{2t^{\frac{1}{2}}}-e^{cz}\operatorname{erfc}\frac{cz+t}{2t^{\frac{1}{2}}}\right]$$
$$+2k_2(\lambda)\left[\operatorname{erfc}\frac{cz-t}{2t^{\frac{1}{2}}}+e^{cz}\operatorname{erfc}\frac{cz+t}{2t^{\frac{1}{2}}}-2\exp\left(\frac{cz}{2}-\frac{t}{4}\right)\operatorname{erfc}\frac{cz}{2t^{\frac{1}{2}}}\right]+O(t^{\frac{3}{2}}), \quad (3.13)$$

where f_2, k_0, k_1 and k_2 are given by (3.6) and $\lambda = e^{-cz}$. For finite $cz/2t^{\frac{1}{2}}$ but small z and t, (3.13) gives

$$G'(z,t) \approx \operatorname{erfc}(cz/2t^{\frac{1}{2}}),$$

which is identical to the solution of the Rayleigh problem. But, displayed in the form (3.12) and (3.13), the functions H^1 and G^1 clearly show the wavelike behaviour of the superposed flow and express the velocity (and hence the vorticity) as diffusion from a source travelling with velocity $c(\nu\Omega)^{\frac{1}{2}}$ away from the rotating disk. These shear-type decaying wave trains are generated by the interaction of the superposed growing Rayleigh layer with the basic axial inflow.

The real value of the substitution (2.12) and the resulting hierarchy of equations (3.2)–(3.5), etc., lies in the fact that they provide detailed information about the way in which the superposed flow develops, grows and affects the whole flow regime. Immediately after the impulsive change in the angular velocity of the disk, a simple Rayleigh shear, of dimensional thickness of order $(\nu T)^{\frac{1}{2}}$, forms in the azimuthal flow. The superposed Rayleigh layer grows and interacts with the von Kármán centrifugal action to induce radial outflow when terms of order $t^{\frac{1}{2}}$ first become significant. This in turn produces axial inflow (due to continuity) at order t^1. Associated with the axial flow within the secondary (growing) layer is normal suction through the edge of this layer, which develops as terms of order $t^{\frac{3}{2}}$ gain importance. At order $t^{\frac{5}{2}}$, the suction at the edge of the secondary layer generates axial flow in the outer flow regime (of the basic steady state). This axial flow is diverted into radial inflow in this region, which affects the azimuthal motion. By the time this happens, the superposed flow varies on two length scales, one increasing with t and the other being the (fixed) length scale of the basic flow. A characteristic length scale of the inner layer is

$$Z = c(\nu\Omega)^{\frac{1}{2}}T+(\nu T)^{\frac{1}{2}}, \quad (3.14)$$

which evidently increases more rapidly than the thickness of the initial Rayleigh layer. But the growth of this layer is inhibited by the basic steady flow. In the time evolution of the von Kármán flow considered by Thiriot (1940), Nigam (1951) and Benton (1966), the initial development is characterized by a similarity solution. No similitude exists in the present case.

4. Approach to the ultimate state

The dominant contributions to H^1 and G^1 as $t \to \infty$ are associated with the singularities of the field functions at $s = 0$ and $-\frac{1}{4}$. Moreover, the final behaviour corresponds to the regions of the complex plane near $s = 0$. We note from (2.16) that $s = 0$ gives $m = 1$. In order to study the approach to the ultimate state, we expand the functions α, f and k in ascending powers of $m-1$ in the form

$$\alpha = c \sum_{n=0}^{\infty} (m-1)^{n-1} \alpha_n, \quad f = \sum_{n=0}^{\infty} (m-1)^{n-1} f_n(\lambda), \quad k = \sum_{n=0}^{\infty} (m-1)^{n-1} k_n(\lambda), \tag{4.1}$$

where $m-1 = [(s+\frac{1}{4})^{\frac{1}{2}} - \frac{1}{2}]$ is small. In this process the transformed functions retain the branch point of the original expressions. A few terms of (4.1) are given by

$$\alpha_0 = 0, \quad f_0 = \tfrac{1}{2}c^2\lambda h', \quad k_0 = \tfrac{1}{2}c^2\lambda g', \tag{4.2}$$

$$\alpha_1 = \tfrac{1}{2}, \quad f_1 = \tfrac{1}{2}c^2 h + c^2(\alpha_2 - \tfrac{1}{2})\lambda h', \quad k_1 = c^2 g + (\alpha_2 - \tfrac{1}{2})\lambda c^2 g', \tag{4.3}$$

$$\left.\begin{aligned} f_2 &= \alpha_2 c^2 h + c^2(\alpha_3 - \alpha_2 + \tfrac{1}{2})\lambda h' + \bar{f}_2, \\ k_2 &= (2\alpha_2 c^2 - \tfrac{1}{2}) g + c^2(\alpha_3 - \alpha_2 + \tfrac{1}{2})\lambda g' + \bar{k}_2, \end{aligned}\right\} \tag{4.4}$$

where $\bar{f}_2$ and $\bar{k}_2$ are given by

$$\lambda^3 \bar{f}_2''' + \lambda^2(2\bar{f}_2'' + \bar{f}_2 h'' + h\bar{f}_2'' - h'\bar{f}_2') + \lambda(h\bar{f}_2' + h'\bar{f}_2) + 4g\bar{k}_2 = \tfrac{1}{2}\lambda^3 h''' - \tfrac{3}{2}\lambda^2 h'' - \lambda h h', \tag{4.5a}$$

$$\lambda^2 \bar{k}_2'' + \lambda(h\bar{k}_2' - \bar{k}_2 h' + \bar{f}_2 g' - g\bar{f}_2') = \tfrac{1}{2}\lambda^2 g'' + 2(g - \lambda g') - \tfrac{1}{2}hg, \tag{4.5b}$$

$$\bar{f}_2(1) = 0, \quad \bar{f}_2'(1) = -\tfrac{1}{2}, \quad \bar{k}_2(1) = (2c^2)^{-1} - 2\alpha_2. \tag{4.5c}$$

The differential set (4.5) enables us to evaluate α_2. But the functions f_2 and k_2 still contain the unknown α_3, which is to be obtained from the differential equations satisfied by f_3 and k_3.

We solve the differential set (4.5), again by the Fettis–Benton method, and write

$$\bar{f}_2 = \sum_{i=1}^{\infty} m_i \lambda^i, \quad \bar{k}_2 = \sum_{i=1}^{\infty} l_i \lambda^i. \tag{4.6}$$

Substituting for $h, g, \bar{f}_2$ and $\bar{k}_2$ from (3.6) and (4.6) into (4.5a, b) and equating the coefficients of different powers of λ to zero, we get the following recursion relations for m_i and l_i:

$$i^2(i-1)\, m_i = \frac{i(i-1)(i-5)\, b_i}{2} - \sum_{j=1}^{i-1} [(i^2 - 3ij + 3j^2)\, m_i b_{i-j} + 4 l_i a_{i-j} + j b_j b_{i-j}], \tag{4.7}$$

$$i(i-1)\, l_i = \frac{(i-1)(i-4)\, a_i}{2} + \sum_{j=1}^{i-1} [(i-2j)(a_j m_{i-j} + l_j b_{i-j}) - \tfrac{1}{2} a_j b_{i-j}]. \tag{4.8}$$

All the coefficients in the power series can be obtained in terms of m_1 and l_1. The boundary conditions (4.5c) are sufficient to determine the three unknowns, namely m_1, l_1 and α_2. Using the values of a_i and b_i tabulated by Benton (1966), the values of m_1, l_1 and α_2 are

$$m_1 = 1{\cdot}307, \quad l_1 = 1{\cdot}239, \quad \alpha_2 = 0{\cdot}126. \tag{4.9}$$

Term-by-term inversion of (2.12) after substituting from (2.15) and (4.1) yields the following large-time solution:

$$\begin{aligned}H^1(z,t) = {}& \frac{H_z^0}{2c}\left[t-(1+\tfrac{1}{2}t)\,\mathrm{erfc}\,\tfrac{1}{2}t^{\frac{1}{2}}+\left(\frac{t}{\pi}\right)^{\frac{1}{2}}e^{-\frac{1}{4}t}+2\alpha_2\right]\\ &-\frac{H_z^0}{4c}\left[(1+t-cz)\,\mathrm{erfc}\,\frac{|cz-t|}{2t^{\frac{1}{2}}}+2\left(\frac{t}{\pi}\right)^{\frac{1}{2}}\exp\left(-\frac{(cz-t)^2}{4t}\right)-e^{cz}\,\mathrm{erfc}\,\frac{cz+t}{2t^{\frac{1}{2}}}\right]\\ &+\tfrac{1}{4}\left(H^0+c+(1-2\alpha_2)\frac{H_z^0}{c}\right)\left[\mathrm{erfc}\,\frac{|cz-t|}{2t^{\frac{1}{2}}}+e^{cz}\,\mathrm{erfc}\,\frac{cz+t}{2t^{\frac{1}{2}}}\right]-\tfrac{1}{2}c+O(t^{-\frac{1}{2}}e^{-\frac{1}{4}t})\\ &\qquad\text{for}\quad cz-t>0,\end{aligned} \tag{4.10a}$$

$$\begin{aligned}H^1(z,t) = {}& \tfrac{1}{2}(H^0+zH_z^0)+\frac{H_z^0}{2c}\left[-\left(1+\frac{t}{2}\right)\mathrm{erfc}\,\frac{t^{\frac{1}{2}}}{2}+\left(\frac{t}{\pi}\right)^{\frac{1}{2}}e^{-\frac{1}{4}t}\right]\\ &+\frac{H_z^0}{4c}\left[(1+t-cz)\,\mathrm{erfc}\,\frac{|cz-t|}{2t^{\frac{1}{2}}}-2\left(\frac{t}{\pi}\right)^{\frac{1}{2}}\exp\left(-\frac{(cz-t)^2}{4t}\right)+e^{cz}\,\mathrm{erfc}\,\frac{cz+t}{2t^{\frac{1}{2}}}\right]\\ &+\frac{1}{4}\left(H^0+c+(1-2\alpha_2)\frac{H_z^0}{c}\right)\left[-\mathrm{erfc}\,\frac{|cz-t|}{2t^{\frac{1}{2}}}+e^{cz}\,\mathrm{erfc}\,\frac{cz+t}{2t^{\frac{1}{2}}}\right]\\ &+O(t^{-\frac{1}{2}}e^{-t/4})\quad\text{for}\quad cz-t<0,\end{aligned} \tag{4.10b}$$

$$\begin{aligned}G^1(z,t) = {}& \frac{G_z^0}{2c}\left[t-(1+\tfrac{1}{2}t)\,\mathrm{erfc}\,\tfrac{1}{2}t^{\frac{1}{2}}+\left(\frac{t}{\pi}\right)^{\frac{1}{2}}e^{-\frac{1}{4}t}+2\alpha_2\right]\\ &-\frac{G_z^0}{4c}\left[(1+t-cz)\,\mathrm{erfc}\,\frac{|cz-t|}{2t^{\frac{1}{2}}}+2\left(\frac{t}{\pi}\right)^{\frac{1}{2}}\exp\left(-\frac{(cz-t)^2}{4t}\right)-e^{cz}\,\mathrm{erfc}\,\frac{cz+t}{2t^{\frac{1}{2}}}\right]\\ &+\frac{1}{2}\left(G^0+(1-2\alpha_2)\frac{G_z^0}{2c}\right)\left[\mathrm{erfc}\,\frac{|cz-t|}{2t^{\frac{1}{2}}}+e^{cz}\,\mathrm{erfc}\,\frac{cz+t}{2t^{\frac{1}{2}}}\right]\\ &+O(t^{-\frac{1}{2}}e^{-\frac{1}{4}t})\quad\text{for}\quad cz-t>0,\end{aligned} \tag{4.11a}$$

$$\begin{aligned}G^1(z,t) = {}& G^0+\tfrac{1}{2}zG_z^0+\frac{G_z^0}{2c}\left[-\left(1+\frac{t}{2}\right)\mathrm{erfc}\,\tfrac{1}{2}t^{\frac{1}{2}}+\left(\frac{t}{\pi}\right)^{\frac{1}{2}}e^{-\frac{1}{4}t}\right]\\ &+\frac{G_z^0}{4c}\left[(1+t-cz)\,\mathrm{erfc}\,\frac{|cz-t|}{2t^{\frac{1}{2}}}-2\left(\frac{t}{\pi}\right)^{\frac{1}{2}}\exp\left(-\frac{(cz-t)^2}{4t}\right)+e^{cz}\,\mathrm{erfc}\,\frac{cz+t}{2t^{\frac{1}{2}}}\right]\\ &+\frac{1}{2}\left(G^0+(1-2\alpha_2)\frac{G_z^0}{2c}\right)\left[-\mathrm{erfc}\,\frac{|cz-t|}{2t^{\frac{1}{2}}}+e^{cz}\,\mathrm{erfc}\,\frac{cz+t}{2t^{\frac{1}{2}}}\right]\\ &+O(t^{-\frac{1}{2}}e^{-\frac{1}{4}t})\quad\text{for}\quad cz-t<0.\end{aligned} \tag{4.11b}$$

It is evident from (4.10) and (4.11) that, after sufficient time has elapsed, the superposed fluid motion varies on two different length scales. The flow within the inner layer, whose characteristic length grows according to (3.14), is now completely permeated by diffused wave trains. The wave front of these shear-type waves moves with velocity $c(\nu\Omega)^{\frac{1}{2}}$. These waves, however, decay through the thickness of the basic azimuthal and radial flow. The superposed flow in the outer region, which extends to the region of the von Kármán state, is primarily generated by the suction induced at the edge of the inner layer. Making $t\to\infty$ in the expressions (4.10) and (4.11) for $H^1(z,t)$ and $G^1(z,t)$, we find that the new steady state, corresponding to the changed angular velocity of the rotating disk, is given by

$$H(z,\infty) = H^0 + \tfrac{1}{2}\epsilon(H^0 + zH_z^0), \quad G(z,\infty) = G^0 + \epsilon(G^0 + \tfrac{1}{2}zG_z^0). \tag{4.12}$$

It is easy to verify that the final state (4.12) is in agreement with the state expected from (2.10) without looking at the transition. The final state of the spin-up is effectively achieved in a time $4/c^2\Omega = 5{\cdot}115\Omega^{-1}$.

Situations similar to the one considered in the present paper have been studied by Greenspan & Howard (1963) and Benton (1966). Greenspan & Howard discussed the spin-up of a viscous fluid confined between two infinite parallel coaxial disks initially in rigid-body rotation. Benton (1966) has found that the ultimate approach to the steady state of the Greenspan & Howard problem is a small amplitude decaying oscillation (of twice the frequency of the rigid-body rotation) about the steady state. Benton (1966) has described the time evolution of the von Kármán flow due to a single rotating disk. His numerical results also indicate a non-monotonic approach to the steady-state values. In analogy with the Greenspan & Howard case, Benton attributes this to small amplitude oscillations of frequency 2Ω about the ultimate state. In the present case also, the approach to the final state is non-monotonic. To demonstrate this we consider the velocity field at the wave front. For $cz = t$, (4.10) and (4.11) give

$$H^1(z,t) - \tfrac{1}{2}(H^0 + zH_z^0) = -\frac{H_z^0}{2c}\left(\frac{t}{\pi}\right)^{\frac{1}{2}} + O(1), \tag{4.13}$$

$$G^1(z,t) - (G^0 + \tfrac{1}{2}zG_z^0) = -\frac{G_z^0}{2c}\left(\frac{t}{\pi}\right)^{\frac{1}{2}} + O(1). \tag{4.14}$$

Since H_z^0 and G_z^0 are negative at large distances, these results clearly indicate that, near the wave front, the large-time values overshoot the steady-state value. Although the non-monotonic behaviour of the fluid motion in the final stages of the spin-up agrees with the results of Greenspan & Howard (1963) and Benton (1966), the solution (4.1) provides no evidence of the oscillatory character of the superposed flow. Such behaviour emerges entirely because of the propagation of shear-type diffusing waves which are generated owing to the interaction of viscous diffusion with the basic axial flow.

The large-time solution (4.1) depends on an expansion about $s = 0$, and hence any singularity which could give rise to a decaying oscillation is precluded from appearing. In order to expose any possible periodic motion, in the next section we investigate briefly the contribution of singularities off the real axis of the complex s plane.

5. A general approach

A general solution of the set (2.18) will be of the form

$$f(s,\lambda) = \frac{1}{L}\sum_{n=1}^{\infty} f_n \lambda^n, \quad k(s,\lambda) = \frac{1}{L}\sum_{n=1}^{\infty} k_n \lambda^n, \tag{5.1}$$

where L, f_n and k_n are polynomials in m. On factorization of L and f_n, the transformed function $s^{-1}\bar{H}(s,\lambda)$ will be a linear combination of terms of the form

$$\frac{\exp[-(s+\frac{1}{4})^{\frac{1}{2}}cz]}{s[\beta+(s+\frac{1}{4})^{\frac{1}{2}}]}\sum_{n=1}^{\infty}\bar{f}_n\exp[-(n-\tfrac{1}{2})cz], \tag{5.2}$$

where $\bar{f}_n$ etc. are constants and β may be real or complex. The inverse of (5.2) is

$$\left[(2\beta-1)\operatorname{erfc}\frac{cz-t}{2t^{\frac{1}{2}}}+(2\beta+1)\operatorname{erfc}\frac{cz+t}{2t^{\frac{1}{2}}}-\beta\exp((\beta+\tfrac{1}{2})cz+\beta^2t-\tfrac{1}{4}t)\operatorname{erfc}\frac{cz+2\beta t}{2t^{\frac{1}{2}}}\right] \times\sum_{n=1}^{\infty}\bar{f}_n e^{-nzc}. \tag{5.3}$$

Terms similar to the first two terms in the above expression have already appeared in the solutions (3.13) and (3.14) and (4.10) and (4.11), and should be interpreted in the same manner. All the possible modes of oscillation of the superposed fluid flow are associated with complex values of β occurring in the third term in (5.3).

When $\beta=\gamma+i\delta$, we use a result of Strand (1965) to expand the error function with complex argument (see appendix and also Chawla 1972). We get

$$\exp[c\beta z+\beta^2t-\tfrac{1}{4}t]\operatorname{erfc}\frac{cz+2\beta t}{2t^{\frac{1}{2}}}$$

$$=\exp[c\gamma z+(\gamma^2-\delta^2-\tfrac{1}{4})t]\,\phi[|cz+2\gamma t|/2t^{\frac{1}{2}},\delta t^{\frac{1}{2}}]\quad\text{for}\quad cz+2\gamma t>0, \tag{5.4a}$$

$$=\exp[c\gamma z+(\gamma^2-\delta^2-\tfrac{1}{4})t]\,[2\exp(-i\delta|cz+2\gamma t|)+\phi(|cz+2\gamma t|/2t^{\frac{1}{2}},\delta t^{\frac{1}{2}})]\quad\text{for}\quad cz+2\gamma t<0. \tag{5.4b}$$

We find that, for a complex β with negative real part, the region of fluid flow between the rotating disk and the moving surface $cz+2\gamma t=0$ can support harmonic waves of wavenumber $c\delta$ and frequency $2|\gamma|\delta$. Continuity considerations immediately suggest that in such a case the fluid column ahead of the wave front $cz+2\gamma t=0$ should be oscillating vertically. This would in turn make the whole of the flow regime oscillate [see (2.12)] about the growing Rayleigh layer and the basic von Kármán state. No oscillatory motion corresponds to β with non-negative real part.

We now study the various modes of oscillation by constructing truncated solutions of the set (2.18). A two-term solution gives

$$\alpha=M_2/L_2=2mc/[c^4m^3(m+1)+(m+2)], \tag{5.5}$$

whereas a four-term solution yields

$$\alpha=M_4/L_4, \tag{5.6}$$

with
$$M_4=24cm[2c^4m^3(m+1)(2m+1)(m+2)(m+3)+3m^6+17m^5+5m^4-28m^3-7m^2+28m+12],$$

$$L_4=24c^4m^3(m+1)(m+2)(m+3)(2m+1)[c^4m^3(m+1)+(m+2)]-(2m^8+5m^7-66m^6-239m^5+58m^4+396m^3-312m^2-816m-288).$$

Choosing only those factors of L which lead to oscillatory motion, we get from (5.5)

$$\beta_{21}=-0{\cdot}283\pm1{\cdot}180i. \tag{5.7}$$

The four-term solution (5.6) modifies this value to

$$\beta_{41} = -0{\cdot}234 \pm 1{\cdot}231i \tag{5.8a}$$

and gives an additional mode

$$\beta_{42} = -0{\cdot}167 \pm 0{\cdot}536i. \tag{5.8b}$$

A six-term expression for L modifies these values further though very slightly and introduces one more mode, with $2|\gamma_{63}|\delta_{63}$ of order 10^{-2}.

Associated with β_{41} are oscillations of frequency $0{\cdot}450\Omega$. But these oscillations die out in a time $0{\cdot}748\Omega^{-1}$, which is much less than the spin-up time ($= 5{\cdot}115\Omega^{-1}$). The second mode, given by β_{42}, corresponds to oscillations of frequency $0{\cdot}140\Omega$, decaying in a time $2{\cdot}509\Omega^{-1}$. The period ($= 45{\cdot}008\Omega^{-1}$) of these oscillations is much larger than the spin-up time. Similarly the periods of all other modes are too large to be able to induce appreciable oscillations before decaying. We conclude that the new von Kármán state is achieved without the fluid undergoing effective oscillatory motion during the spin-up process.

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Appendix

Strand (1965) has shown that, for $x > 0$ and $y \geqslant 0$,

$$\begin{aligned}\operatorname{erfc}(x+iy) &= e^{-2ixy} \sum_{n=0}^{\infty} (xy)^{2n} [\gamma_n(x) - ixy(n+1)\,\gamma_{n+1}(x)] \\ &= e^{-2ixy}\phi(x,y) \quad \text{(say)},\end{aligned} \tag{A 1}$$

where $$\gamma_{n+1}(x) = \frac{2}{(2n+1)\pi^{\frac{1}{2}}}\left[\frac{e^{-x^2}}{(n+1)!\,x^{2n+1}} - \frac{\pi^{\frac{1}{2}}\gamma_n(x)}{n+1}\right] \quad (n = 0, 1, 2, \ldots), \tag{A 2}$$

with $$\gamma_0(x) = \operatorname{erfc} x.$$

Since $\operatorname{erf}[-(x+iy)] = -\operatorname{erf}(x+iy)$ and $\operatorname{erf}(x-iy) = \overline{\operatorname{erf}(x+iy)}$, these cases are also covered by (A 1) and (A 2), but the case $x = 0$ is not. $\phi(x,y)$ is a complex function which tends to zero as $x \to \infty$.

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